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Calculation of combined diffusive and convective mass transfer

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ABSTRACT

The clearance of a dialyzer is calculated under the most general conditions, allowing not only for a mixed diffusive and convective mass transfer, but also for a variation along the membrane of the local ultrafiltration, the membrane permeability and the sieving coefficient. The study is then carried on for the case in which these are all constant, to reach a relatively simple expression for the influence of a low ultrafiltration rate on the clearance. In this study, the permeabilities of the boundary layers on both sides are treated as included in the (equivalent) membrane. In an appendix, the stacking of membranes is studied, giving a general law for the calculation of overall permeabilities of a stack of individual membranes, regarded as one (equivalent) membrane (such as a physical membrane with two boundary layers). Permeability data for boundary layers are quoted from earlier works. In other appendices, the variation of the local ultrafiltration along the dialysis path is studied, as well as its effect on the effective permeability of the membrane.

1. General calculation of the clearance of a dialyzer

A segment of a dialyzer in countercurrent operation is shown in Fig. 1. The effects of boundary layers are included in the membrane. We allow for a variation along the axial coordinate x of both ultrafiltration and membrane permeability data.

In Fig. 1, the following notations are used:

k = permeability factor
L = active length
A = total membrane surface area
S = sieving coefficient
C_b = blood concentration of solute
C_d = dialysate concentration of solute
Q_b = blood flow
Q_d = dialysate flow
q_u = axial density of the ultrafiltration flow
x = axial coordinate from the blood entrance end.

Below, the clearance is denoted by Q_c and the total ultrafiltration by Q_u. Furthermore, $K = kA/L$ and $C_{bL} = C_b(L)$. Other notations are introduced in the text and all are listed at the end of it.

Here, k, S and q_u may be functions of x. Accordingly, C_b C_d, Q_b, Q_d and K are functions of x.

Mass balance yields

$$\begin{cases} -Q_b \dfrac{dC_b}{dx} + q_u C_b = (K+Sq_u)C_b - KC_d, & (1) \\ -Q_d \dfrac{dC_d}{dx} + q_u C_d = (K+Sq_u)C_b - KC_d. & (2) \end{cases}$$

Rearranged:

$$\begin{cases} -Q_b \dfrac{dC_b}{dx} = [K-(1-S)q_u]C_b - KC_d, & (3) \\ -Q_d \dfrac{dC_d}{dx} = (K+Sq_u)C_b - (K+q_u)C_d. & (4) \end{cases}$$

Furthermore:

$$\begin{cases} \dfrac{dQ_b}{dx} = -q_u, \quad Q_b = Q_{bo} - \int_0^x q_u\,dx, & (5) \\ \dfrac{dQ_d}{dx} = -q_u, \quad Q_d = Q_{do} - \int_0^x q_u\,dx = \\ Q_{dL} + Q_u - \int_0^x q_u\,dx, & (6) \end{cases}$$

wherein $Q_{bo} = Q_b(0)$, $Q_{do} = Q_d(0)$ and $Q_{dL} = Q_d(L)$.

A mass balance over the section from x to L gives, if $C_d(L) = 0$ (the case $C_d(L) \neq 0$ is easily handled through superposition of concentrations in the usual, known manner),

$$Q_b C_b - Q_{bL} C_{bL} = Q_d C_d, \quad (7)$$

or

$$C_d = \frac{Q_b C_b - Q_{bL} C_{bL}}{Q_d}. \quad (8)$$

With (3), one then finds

$$-Q_b \frac{dC_b}{dx} = \left[K\left(1 - \frac{Q_b}{Q_d}\right) - (1-S)q_u\right]C_b + \frac{KQ_{bL}}{Q_d} C_{bL}. \quad (9)$$

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One may first solve the corresponding homogeneous equation:

$$-Q_b \frac{dC_b}{dx} = \left[K\left(1-\frac{Q_b}{Q_d}\right) - (1-S)q_u\right]C_b. \quad (10)$$

Introducing

$$f(x) = K\left(\frac{1}{Q_b} - \frac{1}{Q_d}\right) - (1-S)\frac{q_u}{Q_b}, \quad (11)$$

one finds

$$C_b = \beta e^{-\int_0^x f(\eta)\,d\eta}, \quad (12)$$

wherein β is a constant.

Applying the method of «varying the constant», (9) can now be solved. Introducing

$$g(x) = \frac{KQ_{bL}}{Q_b Q_d}, \quad (13)$$

(9) can be written as

$$\frac{dC_b}{dx} = -f(x)C_b - g(x)C_{bL}, \quad (14)$$

with the solution

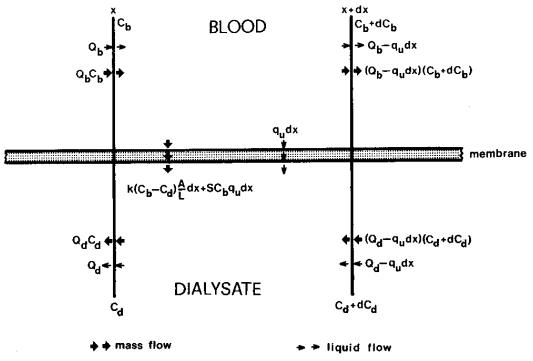
$$C_b = e^{-\int_0^x f(\eta)\,d\eta}\left[C_o - C_{bL}\int_0^x g(\xi)\, e^{\int_0^\xi f(\eta)\,d\eta}\,d\xi\right] \quad (15)$$

Herein, C_o is a constant, which amounts to the blood concentration of the solute at x = O, i.e., $C_o = C_b(O)$. Herewith, C_d is found from equation (7).

For calculating the clearance, one needs the outlet concentration of the solute in the blood, C_{bL}, which is found through putting x = L in (15):

$$C_{bL}\left[e^{\int_0^L f(\eta)\,d\eta} + \int_0^L g(\xi)\, e^{\int_0^\xi f(\eta)\,d\eta}\,d\xi\right] = C_o \quad (16)$$

FIG. 1



By definition, the clearance for zero inlet concentration in the dialysate is

$$Q_c = Q_{bo} - Q_{bL}\frac{C_{bL}}{C_o}, \quad (17)$$

from which, with (16) (after rearrangement):

$$Q_c = Q_{bo} - \frac{Q_{bL}}{\int_0^L [f(\xi) + g(\xi)]\, e^{\int_0^\xi f(\eta)\,d\eta}\,d\xi}. \quad (18)$$

In most cases, q_u is an almost linear function of x, and pressures in blood and dialysate, resp., usually fall almost linearly in the flow direction (cf. Appendix 1). One may therefore put

$$q_u = m - nx, \quad (19)$$

wherein m and n are constants. One then finds, with (5) and (6),

$$Q_b = Q_{bo} - mx + \frac{nx^2}{2}, \quad (20)$$

$$Q_d = Q_{dL} + Q_u - mx + \frac{nx^2}{2}, \quad (21)$$

referred to inlet flows Q_{bo} and Q_{dL}, resp.

Where applicable, the expressions (19) - (21) are, therefore, to be inserted into f(x) and g(x). Furthermore, one generally has (whatever the course of q_u with x): $Q_{bL} = Q_{bo} - Q_u$ and $Q_{do} = Q_{dL} + Q_u$.

Under the condition (A.19), which holds for most dialyzers, one finds from Appendix 1:

$$q_u \approx \frac{Q_u}{L} + \left(\frac{L}{2} - x\right)\frac{U}{L}(\Delta p_b + \Delta p_d), \quad (22)$$

wherein Δp_b and Δp_d are the pressure drops in the blood and dialysate pathways of the membrane section of the device, resp., here taken as approximately constant for various Q_u. This follows from (A.16) and the mean transmembrane pressure

$$\bar{p}_{tm} \approx P_{tmo} - {}^1/_2\,(\Delta p_b + \Delta p_d), \quad (23)$$

wherein p_{tmo} is the transmembrane pressure at x = O, combined with (A.21) and (A.22) in Appendix 1. U is the ultrafiltration coefficient of the membrane:

$$U = \frac{Q_u}{\bar{p}_{tm}}, \quad (24)$$

which in most cases can be treated as a constant (in reality, it varies somewhat with $\bar{p}_{tm}$; for a hollow fiber dialyzer by extremely small amounts and somewhat more for compliant dialyzers).

The contribution from the second term in (22) is in the order of ± 40 ml/h, or ± 0.7 ml/min for typical dialyzers. Integrated over the membrane from x = O to x = L, this second term becomes zero. Therefore, this term may be neglected in comparison with a Q_b of 200 ml/min and a Q_d of 500 ml/min, which flow values constitute the standard operating condition for dialyzer evaluation. One

may therefore write

$$q_u \approx \frac{Q_u}{L}, \tag{25}$$

which renders q_u constant and considerably simplifies the application of (18).

For the application in more general cases, it is advantageous to first calculate

$$\frac{C_o}{C_{bL}} = \int_o^L \left[f(\xi) + g(\xi)\right] \int_e^{\xi} f(\eta)\, d\eta \; d\xi, \tag{26}$$

which follows from (16), and then calculate Q_c from (17). In the case of a constant q_u, one gets the following:

$$Q_b = Q_{bo} - q_u x, \tag{27}$$

$$Q_d = Q_{do} - q_u x, \tag{28}$$

$$f(x) = \frac{K - (1\text{-}S)q_u}{Q_{bo} - q_u x} - \frac{K}{Q_{do} - q_u x}, \tag{29}$$

$$g(x) = \frac{K(Q_{bo} - Q_u)}{Q_{do} - Q_{bo}} \left(\frac{1}{Q_{bo} - q_u x} - \frac{1}{Q_{do} - q_u x}\right). \tag{30}$$

Further limiting to the case of constant K and S, one finds, using $KL = kA$ and $q_u L = Q_u$,

$$\frac{C_o}{C_{bL}} = \frac{\left(1 - \frac{Q_u}{Q_{do}}\right)^{\frac{kA}{Q_u}}}{\left(1 - \frac{Q_u}{Q_{bo}}\right)^{\frac{kA}{Q_u} - 1 + S}} + \frac{kA}{L} \frac{Q_{bo} - Q_u}{Q_{bo} Q_{do}} \int_0^L \frac{\left(1 - \frac{Q_u \xi}{Q_{do} L}\right)^{\frac{kA}{Q_u} - 1}}{\left(1 - \frac{Q_u \xi}{Q_{bo} L}\right)^{\frac{kA}{Q_u} + S}} d\xi. \tag{31}$$

The integral in (31) may also be written as

$$-\frac{Q_u}{L Q_{bo}} \int_0^{\Phi} \varphi^{-(\frac{kA}{Q_u} + S)} \left(1 - \frac{Q_{bo}}{Q_{do}} + \varphi \frac{Q_{bo}}{Q_{do}}\right)^{\frac{kA}{Q_u} - 1} d\varphi, \tag{32}$$

wherein

$$\Phi = 1 - \frac{Q_u}{Q_{bo}}. \tag{33}$$

This integral cannot be generally solved in expressions of elementary functions. For small Q_u, in relation to Q_{bo} and Q_{do}, one may, however, approximate the integral in (31) as

$$\int_e^L \frac{\xi}{L} \left(\frac{kA + SQ_u}{Q_{bo}} - \frac{kA - Q_u}{Q_{do}}\right) d\xi, \tag{34}$$

and, similarly approximating the first term in (31), one finds

$$\frac{C_o}{C_{bL}} \approx e^{kA \frac{Q_{do} - Q_{bo}}{Q_{do} Q_{bo}} - (1\text{-}S) \frac{Q_u}{Q_{bo}}} \left[1 + \frac{kA(Q_{bo} - Q_u)\, e^{Q_u \frac{Q_{bo} + Q_{do}}{Q_{bo} Q_{do}}}}{kA(Q_{do} - Q_{bo}) + Q_u (SQ_{do} + Q_{bo})}\right] - \frac{kA(Q_{bo} - Q_u)}{kA(Q_{do} - Q_{bo}) + Q_u (SQ_{do} + Q_{bo})}. \tag{35}$$

These approximations become exact as $Q_u \to 0$.

Developing into a series in Q_u and keeping only first-order terms, one finds, for $Q_u << Q_b, Q_d$,

$$\frac{C_o}{C_{bL}} (Q_{do} - Q_{bo}) \approx Q_{do}\, e^{kA \frac{Q_{do} - Q_{bo}}{Q_{do} Q_{bo}}} - Q_{bo} + Q_u \left\{1 + e^{kA \frac{Q_{do} - Q_{bo}}{Q_{do} Q_{bo}}} \left[\frac{Q_{bo}}{Q_{do}} - (1\text{-}S)\frac{Q_{do}}{Q_{bo}} - \frac{Q_{bo}}{kA} \frac{SQ_{do} + Q_{bo}}{Q_{do} - Q_{bo}}\right] + \frac{Q_{bo}}{kA} \frac{SQ_{do} + Q_{bo}}{Q_{do} - Q_{bo}}\right\}. \tag{36}$$

The same expression is found through differentiation of (35).

With (36), the variation of C_{bL} with small values of Q_u can be calculated. One herein has to consider the variation of kA with Q_u, which is discussed in Appendix 2. From (A.28), one approximately has

$$kA \approx \frac{2(k_o A)^2}{2k_o A + Q_u S}, \tag{37}$$

wherein k_o is the value of k at $Q_u = 0$. (37) is valid for a

flat membrane arrangement with the surface area A, or for a hollow-fiber dialyzer with the internal surface area A. However, for a hollow-fiber dialyzer, the surface area A in (36) is the actual surface area, which is the logarithmic mean value of the internal and external surface areas (cf. [3] and Appendix 2). Accordingly, a factor expressing surface relations has to be entered in (37) for a hollow-fiber dialyzer — cf. Appendix 2.

The clearance Q_{co} at $Q_u = O$ is found from (17) and (35) (being exact at $Q_u = O$):

$$Q_{co}=Q_{bo}Q_{do}\frac{1-e^{-k_oA\frac{Q_{do}-Q_{bo}}{Q_{do}Q_{bo}}}}{Q_{do}-Q_{bo}\,e^{-k_oA\frac{Q_{do}-Q_{bo}}{Q_{do}\,Q_{bo}}}}, \tag{38}$$

from which one finds

$$k_oA=\frac{Q_{do}\,Q_{bo}}{Q_{do}-Q_{bo}}\ln\left(\frac{Q_{bo}}{Q_{do}}\,\frac{Q_{do}-Q_{co}}{Q_{bo}-Q_{co}}\right). \tag{39}$$

Expanding (37) and the exponential function of kA in series, one then finds for small values of Q_u

$$\Delta\left(\frac{C_o}{C_{bL}}\right)=\frac{C_o}{C_{bL}}-\frac{C_o}{C_{bL}(O)}\approx$$

$$\approx\frac{Q_u}{Q_{do}-Q_{bo}}\left\{\frac{Q_{do}-Q_{co}}{Q_{bo}-Q_{co}}\left[\left(\frac{Q_{bo}}{Q_{do}}\right)^2-1+S\right]+1\right\}-$$

$$-\frac{Q_uQ_{co}\left(S+\frac{Q_{bo}}{Q_{do}}\right)}{Q_{do}(Q_{bo}-Q_{co})\ln\left(\frac{Q_{bo}}{Q_{do}}\,\frac{Q_{do}-Q_{co}}{Q_{bo}-Q_{co}}\right)}-$$

$$-\frac{Q_uS}{2Q_{do}}\,\frac{Q_{do}-Q_{bo}}{Q_{bo}-Q_{co}}, \tag{40}$$

wherein $C_{bL}(O)$ is C_{bL} at $Q_u = O$.

The standard operating condition for a dialyzer is at Q_{bo} = 200 ml/min and Q_{do} = 500 ml/min (rather than Q_{dL} = 500 ml/min, since the suction which creates ultrafiltration is usually generated by means of a controlled dialysate pump placed at the outlet side of the dialyzer). Under this condition, one finds

$$\Delta\left(\frac{C_o}{C_{bL}}\right)\approx\frac{Q_u}{300}\left[\frac{500\text{-}Q_{co}}{200-Q_{co}}(S-0.84)+1\right]-$$

$$-\frac{Q_uQ_{co}(S+0.4)}{500(200-Q_{co})\ln\left(0.4\,\frac{500-Q_{co}}{200-Q_{co}}\right)}-\frac{0.3\,Q_uS}{200-Q_{co}}, \tag{41}$$

if Q_{co} and Q_u are inserted in ml/min.

For small values of Q_u, the clearance according to (17) is

$$Q_c\approx Q_{co}+Q_u\frac{C_{bL}(O)}{C_o}+Q_{bo}\left[\frac{C_{bL}(O)}{C_o}\right]^2\Delta\left(\frac{C_o}{C_{bL}}\right), \tag{42}$$

and, furthermore,

$$\frac{C_{bL}(O)}{C_o}=1-\frac{Q_{co}}{Q_{bo}}. \tag{43}$$

One therefore finds

$$Q_c\approx Q_{co}+Q_u\left(1-\frac{Q_{co}}{Q_{bo}}\right)+$$

$$+Q_{bo}\left(1-\frac{Q_{co}}{Q_{bo}}\right)^2\Delta\left(\frac{C_o}{C_{bL}}\right), \tag{44}$$

wherein $\Delta\left(\frac{C_o}{C_{bL}}\right)$ is found from (40) or (41). With this, one can now calculate the variation af Q_c with Q_u for $Q_u \ll Q_b, Q_d$. For this purpose, one may first define

$$\Delta Q_{c1}=Q_u\left(1-\frac{Q_{co}}{Q_{bo}}\right) \tag{45}$$

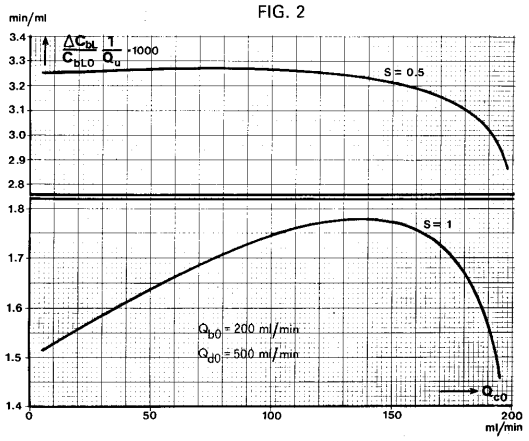
and

$$\Delta Q_{c2}=Q_{bo}\left(1-\frac{Q_{co}}{Q_{bo}}\right)^2\Delta\left(\frac{C_o}{C_{bL}}\right). \tag{46}$$

ΔQ_{c1} is the common estimate for the alteration of Q_c with Q_u, since experimental studies show that C_{bL} remains practically unaltered under ultrafiltration. Nevertheless, ΔQ_{c2} is found to give a significant contribution, as will be seen in the following.

With the relations so found, one can calculate $\Delta C_{bL} = C_{bL}(O)\text{-}C_{bL}$, as well as ΔQ_{c1} and ΔQ_{c2}. The results are shown for S = 0.5 and S = 1 under the standard operating condition in Figs. 2 and 3. In Fig. 2, ΔC_{bL} per unit ultrafiltration (ml/min) is given in ‰ of C_{bL}. In Fig. 3, the negative sign of ΔQ_{c2} is to be noted. It is seen that

FIG. 2



$\Delta C_{bL}/Q_u$ falls in the range of 1-4‰ of C_{bL}, but still ΔQ_{c2} falls between —29% and —66% of ΔQ_{c1} in practical cases. This can be undestood from (45) and 46), in which the factor Q_u in ΔQ_{c1} is much smaller than the factor Q_{bo} in ΔQ_{c2}. Therefore, correction of clearance according to (45) alone is obviously disputable. (Ranges given are for values of S between 0.5 and 1 — smaller values are rarely actual for typical solutes used in dialyzer evaluation).

It is of interest to compare this with measured values, even though measurements usually are to a relatively high extent influenced by limitations in the accuracy of the chemical analysis, since one has to deal with very small changes in the output concentration (vide supra). In spite of a relatively high scatter in individual figures at various Q_u even for the same dialyzer, values of $\Delta Q_c/\Delta Q_u$ estimated by means of linear regression are given in Table 1, as determined at the HRC in Salt Lake City in 1979 for three types of dialyzers. It is seen that (41) and (44) fit these values exactly with plausible values of S for the vitamin B_{12} clearances in all cases, whereas the urea value is fitted only for «Hemoflow C 0.8», but then exactly at the actual value of S for urea convection through Cuprophan, i.e., S = 0.98 (1).

The use of (45) alone leads to much too high values for vitamin B_{12}, but roughly acceptable values for urea. The measured urea values in Table I should, however, be taken with caution, since these are subject not only to scatter to a higher degree than the values for vitamin B_{12}, but also to the influence of a non-ideal distribution of the dialysate flow in the hollow-fiber bundle (especially pronounced in the «RDi» dialyzer) — the latter influence is considerably lower for vitamin B_{12}. The lack of fit for certain urea values is therefore likely to be caused by such other effects on the measured figures. It would, of course be of interest if an accurate study would be performed, requiring a high number of measurements at several ultrafiltration rates for a statistical reduction of the effect of scatter. Measurements for urea here require much higher accuracy in the chemical analysis, since C_{bL} is 5 or 6 times higher for vitamin B_{12} than it is for urea.* This requirement is hampered by the fact that both the Jaffé and the enzymatic methods of analysis are of comparatively low accuracy. Therefore, other methods should be used.

It may be remarked that if the membrane permeability is taken as constant, i.e., if k_o is used in (36) instead of k according to (37) [which corresponds to dropping the last term in (40) and (41)], a poorer fit is reached (except for the dubious urea values) in Table I.

2. A special case

If $Q_{bo} = Q_{do}$, the integral in (31) can be expressed in elementary functions. One finds

$$\frac{C_o}{C_{bL}} = \left(1 + \frac{kA}{Q_u S}\right)\left(1 - \frac{Q_u}{Q_{bo}}\right)^{1-S} - \frac{kA}{Q_u S}\left(1 - \frac{Q_u}{Q_{bo}}\right), \quad (47)$$

if $S \neq 0$. The case $S = 0$ gives (2)

$$\frac{C_o}{C_{bL}} = \left(1 - \frac{Q_u}{Q_{bo}}\right)\left[1 - \frac{kA}{Q_u} \ln\left(1 - \frac{Q_u}{Q_{bo}}\right)\right]. \quad (48)$$

(1) here the surface-relation factor for hollow fibers, mentioned in the text below (37), has been neglected, which is a reasonable approximation.

(2) this is purely theoretical, for the mathematical interest only, since a physical membrane with S = 0 could hardly have a diffusive permeability.

*for the same C_o

TABLE I

Dialyzer	Substance	Measured values		Calculated values: According to (41) and (44)		Calculated values: According to (45)
		Q_{co}	$\frac{\Delta Q_c}{Q_u}$	Measured value fitted at S =	$\frac{\Delta Q_c}{Q_u}$ at S = 0.98	$\frac{\Delta Q_c}{Q_u}$ at any S
Hemoflow C 0.8	urea	120.4	0.252	0.98	0.252	0.398
	vit. B_{12}	23.9	0.494	0.815	—	0.881
Hemoflow C 1.0	urea	160.2	0.219	no fit	0.127	0.199
	vit. B_{12}	31.1	0.416	0.717	—	0.845
Hemoflow C 1.3	urea	164.6	0.142	no fit	0.113	0.177
	vit. B_{12}	41.0	0.497	0.827	—	0.795
Disscap 1.1	urea	161.5	0.230	no fit	0.123	0.193
	vit. B_{12}	28.5	0.427	0.724	—	0.858
RDi	urea	158.7	0.039!	0.2!	0.131	0.207
	vit. B_{12}	41.1	0.385	0.709	—	0.795

Q_{co} is given in ml/min at Q_{bo} = 200 ml/min and Q_{co} = 500 ml/min

The other extreme case S = 1 gives

$$\frac{C_o}{C_{bL}} = 1 + \frac{kA}{Q_{bo}}, \tag{49}$$

independent of Q_u.

3. Calculations of the membrane permeability

Above, the effects of boundary layers were included in the (equivalent) membrane. For practical applications of the above relations, one therefore needs estimates of the permeabilities of those layers, as well as relations for combined (stacked) membranes (since one may regard the physical membrane and the two boundary layers as three membranes stacked one upon the other).

In Appendix 3, a study of combined membranes is undertaken. From this, one can calculate the effective or equivalent overall permeability of any number of membranes stacked together, regarded as one single membrane.

In the actual case, the middle membrane is the physical one and its permeability data are therefore given. The other two «membranes» are equivalent to the effects of the two boundary layers on the blood side and on the dialysate side, resp. For calculation of the overall permeability data, one needs expressions for the latter two «membranes». In [2] a calculation is set up for the dialysate side of hollow fibers (pp. 62-63), which is not carried through to a final expression because of lengthiness. However, the result of such a calculation is found to be

$$k_D = \frac{72\, D_d}{r_2}\,\frac{V}{W}, \tag{50}$$

$$V = (3 - 4t^2 + t^4 + 4 \ln t)^2, \tag{51}$$

$$W = -719 + 1680\, t^2 - 1296\, t^4 + 368\, t^6 - 33\, t^8 - \\ - 120\,(19 - 24\, t^2 + 6\, t^4) \ln t - \\ - 288\,(9 - 4\, t^2) \ln^2 t - 1152 \ln^3 t, \tag{52}$$

wherein k_D is the permeability factor for the dialysate boundary layer and

$$t = r_2 \sqrt{\frac{\pi N}{A_b}}. \tag{53}$$

Here r_2 is the external radius of the wet (swollen) fiber and D_d is the diffusion constant in the dialysate. N is the total number of fibers in the bundle and A_b is the total cross-section area of the bundle (including fiber interspaces).

The blood-side boundary layer permeability is calculated by Babb et al. [4] to be

$$k_B = \frac{D_b}{0.25\, h} \tag{54}$$

for a flat membrane dialyzer, wherein h is the full blood-channel height (from membrane to membrane) and D_b is the diffusion constant in the blood. Klein et al. [5] have adopted the same relation for a hollow-fiber dialyzer, simply putting $h = 2r_1$, where r_1 is the internal radius of the wet fiber. The theory in [2] allows for a more exact calculation of this permeability for hollow fibers. As a result,

$$k_B = \frac{D_b}{2\alpha r_1}, \tag{55}$$

wherein

$$\alpha = 2\left(\frac{1}{p_1^2} - \frac{1}{4w}\right). \tag{56}$$

Herein, p_1 is the first positive root of

$$-p^{10}(0.421880 + w\ 0.0926930)10^{-7} + p^8(0.566862 + \\ + w\ 0.145445)10^{-5} - p^6(0.450304 + w\ 0.144043)10^{-3} + \\ + p^4(0.0182292 + w\ 0.00792101) - p^2(0.25 + w\ 0.1875) + \\ + w = 0 \tag{57}$$

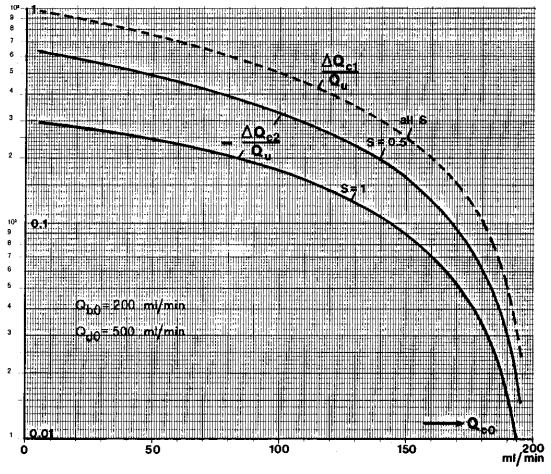
(which is (83) of [2] somewhat corrected and extended). w is here a parameter:

$$w = \frac{k' r_1}{D_b}, \tag{58}$$

wherein k' is the total (combined) permeability of the physical membrane and the dialysate boundary layer (cf. Appendix 3).

The factor α in (55) is shown as a function of w in fig. 4. It is found that $0.229 < \alpha < 0.274$, so that the value 0.25 of [5] is a reasonable approximation.

For a flat membrane dialyzer, [4] states that the permeability of the dialysate boundary layer would be negligible at high values of Q_d, since then Q_c levels off to become almost constant, taken as a function of Q_d. However, as is seen from (38), this is no proof for negligibility of the dialysate boundary layer, since Q_c levels



off in any case, whether there is a significant boundary layer resistance, or not. Furthermore, there is no reason to expect that this permeability would be flow dependent, since the permeability of the blood side boundary layer is not! Instead, for reasons of symmetry, **both** boundary layer permeabilities should depend on geometries (or: **relative** velocity gradients at the membrane), and not on flow values. By analogy to (54), the dialysate boundary layer should therefore have a permeability

$$k_D = \frac{D_d}{\lambda d} \tag{59}$$

in a **flat membrane** dialyzer, wherein d is the height of the dialysate channel. λ is a coefficient which differs from the value in (54) even at laminar dialysate flows, since the dialysate channel has a membrane on one side only, and a solid wall on the other, making for an unsymmetrical concentration distribution across the channel. In many flat membrane dialyzers, the dialysate furthermore flows in a complicated manner inside a membrane support structure (such as a mesh structure or a system of pyramidal protrusions from the solid wall), making for a mixing in the dialysate through multiple local turbulence. An estimate of λ is therefore very difficult to find and no useful literature source is known to the author.

Estimate of boundary layer permeabilities here given are for overall effects, not considering their variations with x. They furthermore apply to the case $Q_u = 0$ and therefore correspond to k_0 in (37) and Appendices 2 and 3. As a first approximation one may put $S = 1$ for boundary layers, but more accurate values should be derived from studies of mass transport in fluid channels with flow components perpendicular to the x-axis — no such study is known to the author.

In Appendix 3, the radial stacking of round, tubular membranes (such as hollow fibers and associated boundary layers) is also discussed. It is found that one has to refer the individual membrane permeabilities to a specific radius, such as the inner radius of the fiber wall. k_B above is already referred to this inner wall radius r_1 in the hollow-fiber case. k_D above, is, however, referred to the outer wall radius r_2 in the hollow-fiber case and can be referred to r_1 through multiplying by r_2/r_1 (cf. Appendix 3).

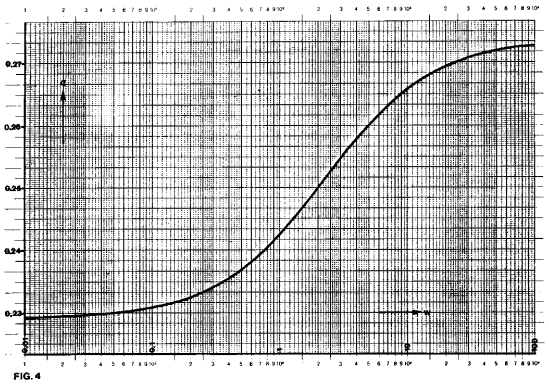


FIG. 4

APPENDIX 1
Study of the ultrafiltration flow density

$q_u(x)$ is the axial ultrafiltration flow density and $q_u(x)L/A$ is the ultrafiltration flow density per unit area, assumed to be uniform, i.e., constant, in a direction perpendicular to x across the membrane.

In most cases, the pressures in the blood and dialysate paths drop linearly in the flow direction at **zero ultrafiltration.** Correspondingly, we can define overall (R) and differential (r) flow resistances:

$$R_b = Lr_b = \frac{\Delta p_b^o}{Q_b^o}, \tag{A.1}$$

$$R_d = Lr_d = \frac{\Delta p_d^o}{Q_d^o}, \tag{A.2}$$

TABLE II - TABLE OF VALUES USED FOR FIGS. 2 AND 3
Q_{bo} = 200 ml/min, Q_{do} = 500 ml/min

Q_{co} ml/min	$\Delta C_{bL}/Q_u C_{bLo} \times 1000$ min/ml		$\Delta Q_{c2}/Q_u$		ΔQ_{c1}
	S = 1	S = 0.5	S = 1	S = 0.5	Q_u
5	1.5149	3.2524	—0.2954	—0.6342	0.975
10	1.5296	3.2548	—0.2906	—0.6784	0.95
20	1.5585	3.2590	—0.2805	—0.5866	0.9
30	1.5865	3.2627	—0.2690	—0.5547	0.85
40	1.6135	3.2658	—0.2582	—0.5225	0.8
50	1.6394	3.2682	—0.2549	—0.4902	0.75
60	1.6640	3.2697	—0.2330	—0.4578	0.7
70	1.6872	3.2703	—0.2193	—0.4251	0.65
80	1.7087	3.2699	—0.2050	—0.3924	0.6
90	1.7282	3.2681	—0.1901	—0.3595	0.55
100	1.7454	3.2649	—0.1745	—0.3265	0.5
110	1.7598	3.2598	—0.1584	—0.2934	0.45
120	1.7708	3.2526	—0.1417	—0.2602	0.4
130	1.7775	3.2427	—0.1244	—0.2270	0.35
140	1.7788	3.2292	—0.1067	—0.1938	0.3
150	1.7729	3.2112	—0.08865	—0.1606	0.25
160	1.7571	3.1867	—0.07028	—0.1275	0.2
170	1.7264	3.1527	—0.05180	—0.09458	0.15
180	1.6709	3.1027	—0.03342	—0.06205	0.1
190	1.5632	3.0192	—0.01563	—0.03019	0.05
195	1.4579	2.9444	—0.007290	—0.01472	0.025

where in Δp_b^o is the pressure drop in the blood path over the membrane and Δp_d^o the corresponding drop on the dialysate side, both at $Q_u = O$. Blood and dialysate flows at $Q_u = O$ are denoted by Q_b^o and Q_d^o, resp. Obviously, from the previous analysis, $Q_b^o = Q_{bo}$ or the entrance blood flow, and $Q_d^o = Q_{do}$ under usual operating conditions [under special conditions one could have $Q_d^o = Q_d$ (L)].

In the general case, one then has a transmembrane pressure

$$P_{tm}(x) = p_{tmo} -$$

$$- r_b \int_o^x Q_b(\xi)\, d\xi - r_d \int_o^x Q_d(\xi)\, d\xi, \qquad (A.3)$$

where $p_{tmo} = p_{tm}(O)$. Hence, from (5) and (6):

$$\begin{cases} p_{tm} = p_{tmo} - x(r_b Q_{bo} + r_d Q_{do}) + \\ \quad + (r_b + r_d) \int_o^x \int_o^\xi q_u(\eta)\, d\eta\, d\xi, & (A.4) \\ \dfrac{dp_{tm}}{dx} = -r_b Q_{bo} - r_d Q_{do} + (r_b + r_d) \int_o^x q_u(\eta)\, d\eta, & (A.5) \\ \dfrac{d^2 p_{tm}}{dx^2} = (r_b + r_d) q_u. & (A.6) \end{cases}$$

q_u is proportional to p_{tm}, and one can put

$$q_u(x) = u^2\, p_{tm}(x), \qquad (A.7)$$

wherein u is a constant. Hence, from (A.6)

$$\frac{d^2 q_u}{dx^2} = u^2 (r_b + r_d)\, q_u, \qquad (A.8)$$

with the solution

$$q_u = a\, e^{xu\sqrt{r_b + r_d}} + b\, e^{-xu\sqrt{r_b + r_d}}, \qquad (A.9)$$

wherein a and b are constants.

At x = O, one then finds, from (A.4) and (A.5),

$$\begin{cases} q_u(O) = u^2 p_{tmo} = a + b, & (A.10) \\ \dfrac{dq_u(O)}{dx} = -u^2(r_b Q_{bo} + r_d Q_{do}) = (a-b)u\sqrt{r_b + r_d}, & (A.11) \end{cases}$$

from which

$$\begin{cases} 2a = u^2 p_{tmo} - u \dfrac{r_b Q_{bo} + r_d Q_{do}}{\sqrt{r_b + r_d}}, & (A.12) \\ 2b = u^2 p_{tmo} + u \dfrac{r_b Q_{bo} + r_d Q_{do}}{\sqrt{r_b + r_d}}. & (A.13) \end{cases}$$

It follows that

$$Q_u = \int_o^L q_u\, dx =$$

$$= \frac{1}{u\sqrt{r_b + r_d}} \left(a\, e^{Lu\sqrt{r_b + r_d}} - b\, e^{-Lu\sqrt{r_b + r_d}} - a + b\right) \qquad (A.14)$$

and, since the mean transmembrane pressure is

$$\bar{p}_{tm} = \frac{1}{L} \int_o^L p_{tm}\, dx = \frac{1}{u^2 L} \int_o^L q_u\, dx, \qquad (A.15)$$

one finds

$$Q_u = u^2 L \bar{p}_{tm} = U \bar{p}_{tm}, \qquad (A.16)$$

or

$$u = \sqrt{\frac{U}{L}}, \qquad (A.17)$$

wherein U is the ultrafiltration coefficient of the membrane.

From (A.9), (A.12) and (A.13), one then can write q_u as

$$q_u = u^2 p_{tmo} \cosh(xu\sqrt{r_b + r_d}) -$$

$$- u \frac{r_b Q_{bo} + r_d Q_{do}}{\sqrt{r_b + r_d}} \sinh(xu\sqrt{r_b + r_d}). \qquad (A.18)$$

From this, it follows:

1. small values of Q_u in relation to Q_b and Q_d, are possible only if

   $$Lu\sqrt{r_b + r_d} = \sqrt{U(R_b + R_d)} \ll 1 \qquad (A.19)$$

   (assuming that the operating conditions are such that $q_u \geqq O$ for all x, since where $q_u < O$ dialysate could otherwise be infused into the blood in case of a membrane leak), otherwise, pressure drops alone make for a high ultrafiltration,
2. under this condition, q_u is an approximately linear function of x at all ultrafiltration values.

An evaluation of existing membranes at typical dialyzer pressure drops indicates that the condition (A.19) is nearly always fulfilled (except, e.g., for long dialyzers with the Membrana HDF Cuprophan membrane at more or less original permeability, i.e., without substantial permeability losses in the dialyzer manufacturing procedure). Therefore, one can in most cases assume a nearly linear function $q_u(x)$ for dialyzers, but usually not for hemofilters and plasmapheresis filters.

Cases in which (A.19) does not hold, are such in which the pressure drops alone produce a comparatively high ultrafiltration. If (A.19) holds, a high ultrafiltration can, of course, be generated by means of a sufficient p_{tmo}; even then, $q_u(x)$ is a nearly linear function.

When $q_u(x)$ is nearly linear, one may write

$$q_u \approx m - nx, \qquad (A.20)$$

in which, from (A.18), (A.17), (A.1) and (A.2),

$$m = \frac{U p_{tmo}}{L}, \qquad (A.21)$$

$$n = \frac{U}{L}(r_b Q_{bo} + r_d Q_{do}) \approx \frac{U}{L^2}(\Delta p_b + \Delta p_d), \qquad (A.22)$$

wherein Δp_b and Δp_d are the actual pressure drops.

Typical values of $\sqrt{U(R_b+R_d)}$ for common dialyzers are in the order of 0.1. Common constructions with high-flux membranes may have values up to an order of 0.25. Hemofilters and plasmapheresis filters have still higher values.

APPENDIX 2
The membrane permeability with and without ultrafiltration

As shown in [1], the total solute flux J_s through a flat membrane under ultrafiltration may be written as (using notations of the preceeding analysis)

$$J_s = k_o(C_1—C_2) + \frac{q_uL}{A} S\bar{C}, \tag{A.23}$$

wherein

$$\bar{C} = C_1 — (C_1—C_2)\left(\frac{1}{\Theta} — \frac{1}{e^{\Theta}—1}\right), \tag{A.24}$$

in which

$$\Theta = \frac{q_uLS}{Ak_o}. \tag{A.25}$$

C_1 is here the concentration at the inside of the membrane and C_2 at the outside, in relation to the direction of J_s — not including boundary layers.

Rearranging (A.23), one finds

$$J_s = k(C_1—C_2) + \frac{q_uL}{A} SC_1, \tag{A.26}$$

wherein, for a flat membrane,

$$k = \frac{q_uLS}{\dfrac{q_uLS}{A(e^{\frac{Ak_o}{}} — 1)}} \tag{A.27}$$

is the actual diffusion permeability factor at the inside of the membrane (cf. Appendix 3).

For small values of q_u, one finds

$$k \approx \frac{2k_o^2A}{2k_oA + q_uLS}, \tag{A.28}$$

which approaches k_o as $q_u \to 0$.

The expression (A.26) appears physically more appropriate than (A.23), since $k(C_1—C_2)$ is the portion of the solute extracted from the blood through diffusion, and q_uLSC_1/A the portion extracted through convection. Therefore, ultrafiltration reduces the effective diffusive permeability (due to reduction of the concentration gradient), but at the same time adds even more through convection — as seen from the blood side. Inside the membrane, the partition between diffusive and convective portions is gradually shifted.

From [2] (where the study is done for S = 1), one finds for a hollow fiber, introducing the sieving coefficient S,

$$k = \frac{q_uLS}{A_1} \frac{1}{\dfrac{q_uLSr_1}{A_1D_w}\left(\dfrac{r_2}{r_1}\right) — 1}, \tag{A.29}$$

wherein A_1 is the inner surface area of the fiber bundle and D_w the apparent diffusion coefficient in the fiber wall (as determined from external surface concentrations). r_1 is the inner and r_2 the outer radius of the wet fiber.

At $q_u = O$, the permeability fator is, according to [2],

$$k_o = \frac{D_w}{r_1 \ln\dfrac{r_2}{r_1}}. \tag{A.30}$$

Inserting this in (A.29), one finds

$$k = \frac{q_uLS}{\dfrac{q_uLS}{A_1k_o}(e^{\frac{A_1k_o}{}} — 1)}. \tag{A.31}$$

As $r_1 \to \infty$, this expression approaches (A.27), since then $A_1 \to A$ (vide infra); at the same time, k_o approaches

$$\lim_{r_1 \to \infty} k_o = \frac{D_w}{\delta}, \tag{A.32}$$

wherein $\delta = r_2—r_1$ is the membrane thickness. (A.32) is the relation for a flat membrane.

As is shown in [3], the inner surface area A_1 of the fiber bundle relates to its true surface area A as

$$A_1 = 2N\pi r_1L. \tag{A.33}$$

$$A = 2N\pi r_mL, \tag{A.34}$$

$$r_m = \frac{r_2 — r_1}{\ln\dfrac{r_2}{r_1}}, \tag{A.35}$$

$$\because A_1 = \frac{Ar_1}{r_2—r_1} \ln\frac{r_2}{r_1}, \tag{A.36}$$

wherein N is the number of fibers in the bundle.

APPENDIX 3
Combination of membranes

In the case of zero ultrafiltration, the combined effect of stacked membranes is easily found. The total diffusion resistance (the inverted total permeability factor) is simply the sum of the individual diffusion resistances. There-

fore, the combined effect of one membrane and the boundary layers (treated as membranes) is found by means of a simple process of superposition.

In the presence of ultrafiltration, things become a bit more complicated. Nevertheless, a kind of superposition process can be devised.

In Appendix 2, the solute flux J_s was written as

$$J_s = k(C_1 - C_2) + \frac{q_u L}{A} SC_1, \tag{A.26}$$

wherein the diffusive portion is expressed as seen from the blood side. Analogously, one may rearrange to

$$J_s = k_d(C_1 - C_2) + \frac{q_u L}{A} SC_2, \tag{A.37}$$

wherein the diffusive portion is expressed as seen from the dialysate side. One here finds

$$k_d = k\, e^{\frac{q_u LS}{Ak_o}}, \tag{A.38}$$

with k according to (A.27), and notes that

$$k_d - k = \frac{q_u LS}{A}, \tag{A.39}$$

i.e., the difference amounts to the convective permeability factor, as seen from either side.

One may also rearrange as

$$J_s = C_1 k_d - C_2 k, \tag{A.40}$$

which expresses a kind of superposition of the diffusive contributions from the respective other side. The contribution from the blood side on the dialysate side is then expressed through $C_1 k_d$ (as if C_2 were zero), and the contribution from the dialysate side on the blood side is expressed through $-C_2 k$ (as if C_1 were zero). This can be developed into a superposition principle for stacked membranes. The influence of convection is in (A.40) implicitly included according to (A.39).

If two flat membranes are stacked together, one can calculate the concentration C_c at their common contact surface (interface) by means of mass balance, equating the output solute flux from the one membrane with the input solute flux to the next, at that surface. This way, one finds the overall diffusive permeabilities

$$k = \frac{k_1 k_2}{k_1 + k_{2d}}, \tag{A.41}$$

and

$$k_d = \frac{k_{1d} k_{2d}}{k_1 + k_{2d}}, \tag{A.42}$$

where, again, sieving coefficients are implicitly included. Indices 1 and 2 denote the individual membranes as numbered from the blood side.

If ν membranes are stacked, numbered 1, ...,ν from the blood side, the following formulae can be shown to be valid by means of induction (treating the stack from 1 to $\nu-1$ as one membrane and membrane ν as the other):

$$k = \frac{1}{\Sigma} \prod_{j=1}^{\nu} k_j, \tag{A.43}$$

$$k_d = \frac{1}{\Sigma} \prod_{j=1}^{\nu} k_{jd}, \tag{A.44}$$

wherein

$$\Sigma = \prod_{j=1}^{\nu-1} k_j + \sum_{i=1}^{\nu-1} \left(\prod_{j=1}^{\nu-i-1} k_j \prod_{j=\nu-i+1}^{\nu} k_{jd} \right) =$$

$$= \sum_{i=1}^{\nu} \left(\prod_{j=1}^{i-1} k_j \prod_{j=i+1}^{\nu} k_{jd} \right), \tag{A.45}$$

with the convention

$$\prod_{j=\mu+1}^{\mu} z_j = 1 \tag{A.46}$$

for all μ (especially for $\mu = 0$ and $\mu = \nu$) and any z_j.

From this, one finds, with (A.39),

$$k_d - k = \frac{1}{\Sigma} \left(\prod_{j=1}^{\nu} k_{jd} - \prod_{j=1}^{\nu} k_j \right) = \frac{q_u LS}{A} \tag{A.47}$$

for the total sieving coefficient S.

With (A.39) for the individual membranes, the difference between the products in (A.47) can also be written as

$$\prod_{j=1}^{\nu} k_{jd} - \prod_{j=1}^{\nu} k_j =$$

$$= \frac{q_u L}{A} \sum_{i=1}^{\nu} \left(\prod_{j=1}^{i-1} k_j \prod_{j=i+1}^{\nu} k_{jd}\, S_i \right), \tag{A.48}$$

so that one finds

$$S = \frac{1}{\Sigma} \sum_{i=1}^{\nu} \left(\prod_{j=1}^{i-1} k_j \prod_{j=i+1}^{\nu} k_{jd}\, S_i \right). \tag{A.49}$$

Applying (A.38) in an analogous manner, one finds

$$\frac{k_d}{k} = e^{\frac{q_u L}{A} \sum_{j=1}^{\nu} \frac{S_j}{k_{oj}}} = e^{\frac{q_u LS'}{Ak_o}} \tag{A.50}$$

so that

$$\frac{S'}{k_o} = \sum_{j=1}^{\nu} \frac{S_j}{k_{oj}}. \tag{A.51}$$

Hence, for $q_u \neq 0$, $S' \neq S$ in the general case. They become equal in the limit as $q_u \to 0$. Therefore, we can approximate $S \approx S'$ for small q_u, from which we conclude that (A.27) and (A.28) can be approximately applied for the total parameters of the stack of membranes, i.e., for the equivalent parameters, taken as one membrane.

Hence for small values of Q_u we can use (A.51) and (A.27) or (A.28) with

$$\frac{1}{k_o} = \sum_{j=1}^{\nu} \frac{1}{k_{oj}}, \tag{A.52}$$

which is the relation for the total permeability factor at zero ultrafiltration.

In the case of tubular membranes stacked radially, things are a bit more complicated still, since the areas of the contact surfaces are no more the same. The solute flux J_{sr1}, referred to the inner radius r_1 of the tubular membrane, is, according to Appendix 2,

$$J_{sr1} = k(C_1 - C_2) + \frac{q_u L}{A} S C_1 = k_d C_1 - k C_2, \tag{A.53}$$

with k according to (A.31). From this one finds

$$k_d = k e^{\frac{q_u L S}{A_1 k_o}} \tag{A.54}$$

and

$$k_d - k = \frac{q_u L S}{A_1}, \tag{A.55}$$

wherein k and k_d are both referred to r_1.

The solute flux at any other radius in the membrane is

$$J_{sr} = \frac{r_1}{r} J_{sr1}, \tag{A.56}$$

as follows from the law of continuity. The diffusive permeabilities change accordingly and become, as referred to r,

$$k_r = \frac{r_1}{r} k, \tag{A.57}$$

$$k_{dr} = \frac{r_1}{r} k_d. \tag{A.58}$$

If one stacks ν membranes radially, having the inner radii $r_1, r_2, \ldots r_\nu$, one finds in the same manner as above:

$$k r_1 = \frac{1}{\Sigma} \prod_{j=j}^{\nu} k_j r_j, \tag{A.59}$$

$$k_d r_1 = \frac{1}{\Sigma} \prod_{j=1}^{\nu} k_{jd} r_j, \tag{A.60}$$

and

$$\Sigma = \sum_{i=1}^{\nu} \left(\prod_{j=1}^{i-1} k_j r_j \prod_{j=i+1}^{\nu} k_{jd} r_j \right) \tag{A.61}$$

with the convention (A.46). Furthermore:

$$S r_1 = \frac{1}{\Sigma} \sum_{i=1}^{\nu} \left(S_i r_i \prod_{j=1}^{i-1} k_j r_j \prod_{j=i+1}^{\nu} k_{jd} r_j \right), \tag{A.62}$$

$$\frac{1}{k_o r_1} = \sum_{j=1}^{\nu} \frac{1}{k_{oj} r_j} \tag{A.63}$$

whereas (A.51) remains unchanged. Again, for small Q_u, one may as an approximation use (A.51) and (A.27) or (A.28) with (A.63).

In (A.59) — (A.63) and in (A.51) for tubular membranes, k_j and k_{jd} are always referred to r_j.

Remark concerning modelling of mass transport in membranes and related experimental studies

It seems that equation (A.40) should be in a suitable general form for description of mass transport through membranes. Various studies have been published, such as in [1] and [6], giving different results for $\bar{C}$ in (A.23). Equation (A.40) applies in both cases, but with different flow dependencies for k and k_d, as functions of the ultrafiltration flow. It should be possible to separately measure k and k_d (or to separate them out from sets of measurements), using suitable experimental arrangements, as functions of q_u. In this way, theories on transport in membranes could be experimentally tested in a relatively strict manner.

Note

In physical membranes, the concentration usually suddenly jumps from the outside value in the immersing liquid just at the membrane surface to a different inside value in the membrane, just inside the surface. However, one may calculate with equivalent values of concentrations in the membrane, neglecting such jumps and equating outside and inside concentrations at the membrane surface. As a result, apparent values of diffusion constants and permeabilities apply, relating to concentration values outside the membrane, just at its surfaces. Concentrations, permeabilities and diffusion constants above which pertain to membranes are to be understood as such equivalent or apparent values. It is anyway the apparent permeability which is measured in the first place, when evaluating membranes — true permeabilities have to be estimated from estimates of inside concentrations, usually difficult to measure.

Notations

a = a constant
A = total membrane area
A_1 = inner membrane area of hollow-fiber bundle
A_b = total cross-section area of hollow-fiber bundle
b = constant
$\bar{C}$ = mean concentration in membrane
C_b = blood concentration in solute
C_{bL} = C_b at x = L
$C_{bL}(O)$ = C_{bL} at Q_u = O
C_d = dialysate concentration of solute
C_o = C_b at x = O
C_1 = concentration at inside of membrane under solute flux J_s

C_2 = concentration on outside of membrane under solute flux J_s
d = dialysate channel height
D_b = diffusion constant in blood
D_d = diffusion constant in dialysate
D_w = apparent diffusion constant in hollow-fiber wall
f = function defined by (11)
g = function defined by (13)
h = blood channel height
i = summation index
j = multiplication and summation index
J_s = solute flux through membrane
J_{sr} = solute flux through hollow-fiber wall at radius r
J_{sr1} = solute flux through hollow-fiber wall at radius r_1
k = permeability factor as seen from blood side of membrane
k' = combined permeability factor for membrane and dialysate boundary layer
k_B = permeability factor for blood boundary layer
k_d = permeability factor as seen from dialysate side of membrane
k_D = permeability factor for dialysate boundary layer
k_{dr} = k_d referred to radius r in hollow fiber
$k_{di,j}$ = k_d of membrane i or j
$k_{i,j}$ = k of membrane i or j
k_r = k referred to radius r in hollow fiber
k_o = k at $Q_u = O$
k_{oj} = k_o of membrane j
K = kA/L
L = active length of dialyzer membrane
m = a constant
n = a constant
N = number of fibers in hollow-fiber bundle
p = variable in (57)
p_{tm} = transmembrane pressure
$\bar{p}_{tm}$ = mean (over x) transmembrane pressure
p_{tmo} = p_{tm} at x = O
p_1 = first positive root of equation (57)
q_u = axial density of ultrafiltration flow
Q_b = blood flow
Q_b^o = Q_b at $Q_u = O$
Q_{bo} = Q_b at x = O
Q_{bL} = Q_b at x = L
Q_c = dialyzer clearance
Q_d = dialysate flow
Q_d^o = Q_d at $Q_u = O$
Q_{do} = Q_d at x = O
Q_{dL} = Q_d at x = L
Q_u = ultrafiltration flow
r = radius in hollow fiber (from center)
r_b = differential blood-path flow resistance
r_d = differential dialysate-path flow resistance
r_m = mean radius of fiber wall
r_1 = inner radius of fiber wall
r_2 = outer radius of fiber wall
R_b = blood-path flow resistance
R_d = dialysate-path flow resistance
S = sieving coefficient
$S_{i,j}$ = S of membrane i or j
S' = a kind of sieving coefficient defined by (A.50)
t = parameter defined by (53)
u = factor in (A.7)
U = ultrafiltration coefficient of membrane
V = function defined by (51)
w = parameter defined by (58)
W = function defined by (52)
x = axial coordinate, i.e., coordinate in blood-flow direction along membrane
z_i = variable in (A.46)

α = parameter defined by (56)
β = a constant
δ = membrane thickness
Δ = change or deviation in subsequent quantity
Δp_b = pressure drop over L in blood path
Δp_b^o = Δp_b at $Q_u = O$
Δp_d = pressure drop over L in dialysate path
Δp_d^o = Δp_d at $Q_u = O$
η = integration variable
Θ = parameter defined by (A.25)
λ = coefficient in (59)
μ = parameter in (A.46)
ν = number of stacked membranes
ξ = integration variable
Σ = sum defined by (A.45) for flat membranes and by (A.61) for hollow-fiber membranes
φ = integration variable
Φ = value defined by (33)

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